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On the Geometry of Sculptured Surface Machining

Johannes Wallner and Helmut Pottmann

Abstract. We present geometric aspects of sculptured surface machining. Several possible configuration manifolds of tool positions relative to a workpiece are investigated under different aspects: the degree of freedom of the motion of the tool, the correspondence between the contact point and the tool position, and the presence or absence of unwanted collisions between tool and workpiece.

§1. Introduction

In the past decades, strong research efforts have been devoted to developing the mathematical fundamentals and efficient algorithms for the representation of free-form surfaces in CAD/CAM systems. However, just a few contributions address *manufacturing* of sculptured surfaces, although there are appealing and practically important open problems in this area.

Geometrical problems in this area include the following: If two surfaces touch each other at a point, such as a milling-tool and a free-form surface which is to be manufactured, does the curvature of the surfaces force them to intersect arbitrarily near the contact point (= the local collision problem)? Given a free-form surface and a milling tool, is the tool able to move such that its envelope during the motion is the free-form surface? Is the tool able to do this while moving only by translations (3-axis milling), or do we need more flexibility (5-axis milling)? Which relative tool position achieves best surface quality (=tool positioning)? How can we decompose the theoretically two-parameter motion of the milling tool by a series of one-parameter motions such that e.g. manufacturing time is minimal (=motion planning)?

A survey of mathematical fundamentals on NC machining of sculptured surfaces is given in [1,14,19]. Tool selection, motion planning and local interference checking for 3-axis and 5-axis machining has been studied in [2,9,10,11, 12,13,25]. Three-axis machining (general offsets and Minkowski addition) are considered by [15,21,23]. A configuration space has been defined in [3,24].

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There is a variety of contributions using Computer Graphics techniques such as visibility algorithms [5,7,22]. There the cutter shaft is shrunk to its axis, and simultaneously the design surface is replaced by an appropriate offset surface. Collision checking is thus transformed to a visibility test (5-axis machining requires a modification of the method).

This paper is organized as follows: In Section 2 we briefly show some concepts of elementary differential geometry. Section 3 sums up some results concerning the local contact situation. Section 4 investigates configuration manifolds of motions constrained in several ways and describes the possible infinitesimal motions of a milling-tool in a contact position. In Section 5 we study the dependency of the contact *point* of the contact *position*. Finally Section 6 features global statements about the absence of unwanted collisions under certain circumstances.

§2. Differential Geometry

We first give a short description of some aspects of curvature theory of 2-surfaces in Euclidean three-space (cf. [4,17,20]).

2.1. Oriented surfaces and their first and second fundamental forms

Consider a regular smooth surface given by the parametrization $f = f(u)$, where $x = (x_1, x_2, x_3)$ is a point in Euclidean $\mathbb{R}^3$ and $u = (u_1, u_2)$ ranges in some open planar domain D. We assume that f is twice continuously differentiable in order to be able to define curvatures.

The differential df of f maps a tangent vector $v = (v_1, v_2)$ attached to the point u, to the vector $d_u f(v) = \frac{d}{dt}|_{t=0} f(u+tv)$, which is computed by $d_u f(v) = x_{,1}(u)v_1 + x_{,2}(u)v_2$, where the symbols $x_{,1}$ and $x_{,2}$ mean differentiation with respect to the first and second variable.

The function $n = (x_{,1} \times x_{,2})/\|x_{,1} \times x_{,2}\|$ is the surface unit normal vector. The symmetric bilinear forms $g_u(v,w) = d_u f(v) \cdot d_u f(w)$ and $h_u(v,w) = -dn(v) \cdot w$ are called the first and second fundamental forms of f. With $g_{ij} = x_{,i} \cdot x_{,j}$ and $h_{ij} = -n_{,i} \cdot x_{,j} = n \cdot x_{,ij}$ we have $g(v,w) = \sum g_{ij} v_i w_j$ and $h(v,w) = \sum h_{ij} v_i w_j$.

2.2. The Dupin indicatrix and Meusnier's theorem

If f is the parametrization of a surface, and $u = u(t)$ is a curve in its parameter domain D, then $c(t) = f(u(t))$ is a curve contained in the surface $f(D)$. The curve's tangent $\dot{c}(t) = df(\dot{u}(t))$ is contained in the surface's tangent plane. Its second derivative vector is split into three components:

$$\ddot{c} = \|\dot{c}\| \left(\kappa_n n + \alpha \dot{c} + \kappa_g b\right),$$

where n is the normal vector evaluated at $u(t)$ and b is the curve's normal in the tangent plane. The coefficients κ_g and κ_n are the geodesic curvature and normal curvature of the curve, respectively. The following theorem states the perhaps unexpected fact that the normal curvature is dependent only on the direction $\dot{c}$, and we can therefore speak of the normal curvature of a surface tangent.

Theorem 1. *(Meusnier) The normal curvature of the curve c is computed by $\kappa_n = \kappa_n(\dot{u}) = h(\dot{u}, \dot{u})/g(\dot{u}, \dot{u})$.*

Evaluate the matrix product $(g_{ij})^{-1}h_{ij}$ at (u_1, u_2), and compute its two linearly independent eigenvectors v', v'' which correspond to eigenvalues κ_1, κ_2. Then $df(v')$, $df(v'')$ are orthogonal and define the two principal surface tangents at u.

Theorem 2. *(Euler) Assume that w', w'' are unit vectors parallel to the principal surface tangents at u. If $df(v) = \cos\phi w' + \sin\phi w''$, then $\kappa_n(v) = \cos^2\phi \cdot \kappa_1 + \sin^2\phi \cdot \kappa_2$. The polar diagrams of $1/\sqrt{\kappa_n}$ and $1/\sqrt{-\kappa_n}$ in the tangent plane (the oriented* Dupin indicatrices *i_+, i_-) are possibly void or singular conic sections centered in the origin.*

Surface points are called elliptic, if $\kappa_1\kappa_2 > 0$, hyperbolic if $\kappa_1\kappa_2 < 0$, flat if $\kappa_1 = \kappa_2 = 0$, and parabolic in the remaining cases.

2.3. Euclidean displacements and infinitesimal motions

A Euclidean displacement $g : x \in \mathbb{R}^3 \mapsto g(x) \in \mathbb{R}^3$ may be written in the form $x \mapsto M \cdot x + v$, where M is an orthogonal matrix of determinant 1. A one-parameter family $g(t) = (v(t), M(t))$ of Euclidean displacements, (= a path of Euclidean motions, or a smooth Euclidean motion) has in all of its instants an infinitesimal motion, which is determined by the velocity vectors $d(g(t))(x)/dt = \dot{v}(t) + \dot{M}(t) \cdot x$ of all points. If an infinitesimal motion coincides with the velocity field of a smooth rotation about an axis, it is called an infinitesimal rotation. The definition of infinitesimal translation and infinitesimal helical motion is analogous.

There is a linear space of infinitesimal motions. It is further well known that all infinitesimal motions (i.e, all velocity fields of smooth motions) can be written in the form

$$\dot{x} = \bar{c} + c \times x.$$

The condition $c \cdot \bar{c} = 0$ characterizes infinitesimal rotations, $c = 0$ characterizes infinitesimal translations, and $c \neq 0$, $c \cdot \bar{c} \neq 0$ characterizes infinitesimal helical motions. We briefly write $(c, \bar{c}) \in \mathbb{R}^6$ to denote an infinitesimal motion. An infinitesimal rotation whose axis is $a + [b]$ then has the form $\lambda(b, a \times b)$ (the symbol $[b]$ denotes all multiples of the vector b). The translation $\dot{x} = \bar{c}$ has the form $(0, \bar{c})$. If we have to consider the coordinates of $c, \bar{c}$ in some coordinate system, we always write $c = (c_{01}, c_{02}, c_{03})$ and $\bar{c} = (c_{23}, c_{31}, c_{12})$.

2.4. Ruled surfaces

If $p(t)$, $v(t)$ are two curves with $v \neq 0$, then $f(u_1, u_2) = p(u_1) + u_2 v(u_1)$ is a surface whose parameter lines $u_1 =$ const are straight lines, which will be denoted by $l(u_1)$. Such a surface is called a ruled surface.

We need the following well known results concerning the first order differential properties of ruled surfaces in Euclidean space: there is an orthonormal frame $(q; e_1, e_2, e_3)$, dependent on a parameter t, and a smooth function $u_1(t)$

such that $q(t)$ is on the ruling $l(u_1(t))$, $e_1(t)$ is parallel to it, and $e_2(t)$ is tangent to the surface at $q(t)$, and which can be chosen such that its infinitesimal motion at $t = 0$ is a helical motion $(c, \bar{c})$ which can be computed by the following formulae:

$$\bar{p} := p \times v \quad s := \frac{1}{\|p \times \dot{p}\|^2} (\det(p, \dot{p}, \dot{\bar{p}})p + \det(p, \bar{p}, \dot{p})\dot{p} + (\bar{p} \cdot \dot{p})(p \times \dot{p}))$$
$$b = \left.\frac{(\dot{p} \times v) \times v}{\|(\dot{p} \times v) \times v\|}\right|_{t=0}, \quad \delta = \left.\frac{(\dot{p}\dot{\bar{p}})p^2}{(p \times \dot{p})^2}\right|_{t=0}, \quad c = \delta b, \quad \bar{c} = b - c \times s(0).$$

The first order differential invariant δ is called the distribution parameter of the ruled surface, and the point $s(t)$ is called its striction point.

2.5. Line congruences

A smooth line congruence $\mathcal{K}$ is a smooth two-parameter family of straight lines $l(u_1, u_2)$ in Euclidean three-space. It may be parametrized by two 'surfaces' $p(u_1, u_2)$ and $v(u_1, u_2)$, where $v(u_1, u_2) \neq 0$. The line $l(u_1, u_2)$ then is $p(u_1, u_2) + [v(u_1, u_2)]$. The choice of a curve $(u_1(t), u_2(t))$ in the parameter domain gives a ruled surface $l(u_1(t), u_2(t))$.

Definition. *A line congruence $\mathcal{K}$ is* regular *at a line l if the six-tuples $(p, p \times v)$, $(p_{,1}, p \times v_{,1} + p_{,1} \times v)$, $(p_{,2}, p \times v_{,2} + p_{,2} \times v)$ are linearly independent.*

The meaning of this definition is that the lines of the congruence actually change infinitesimally if we move infinitesimally in the parameter domain. We will always assume that $\mathcal{K}$ is regular.

It is well known that the infinitesimal properties of first order of a line l within $\mathcal{K}$ are like those of a *linear* line congruence $\mathcal{K}'$, which is called the *tangent* to $\mathcal{K}$. There are the following possibilities for $\mathcal{K}'$:

- $\mathcal{K}'$ is the set of lines which intersect two lines k', k'' (= a hyperbolic linear congruence with axes k', k'').
- $\mathcal{K}'$ is the set of lines whose complex extensions intersect two conjugate complex lines k', k'' (= an elliptic linear congruence). This set of lines is the affine image of the set of lines which join the points $(x, y, 0)$ and $(x \cos\phi - y \sin\phi, x \sin\phi + y \cos\phi, 1)$, $0 \leq \phi < 2\pi$.
- $\mathcal{K}'$ is the set of lines tangent to a ruled quadric in the points of one of its rulings k. (= a parabolic linear congruence). The line k is also called the axis of $\mathcal{K}'$.
- $\mathcal{K}'$ is a bundle of lines.

In all cases, lines at infinity are allowed (but in the elliptic case they do not occur). These four types of lines l in a congruence are accordingly called hyperbolic, elliptic, parabolic, and degenerate.

In the hyperbolic and parabolic case, the points of l contained in an axis are called focal points, in the degerenate case the bundle vertex is also called a focal point.

There is also a connection between infinitesimal motions and linear line congruences: a four-dimensional subspace of the space of infinitesimal motions always contains infinitesimal rotations or translations, whose axes form a linear line congruence (here we assign to the infinitesimal translation $\dot{x} = \bar{c}$ the axis at infinity which is orthogonal to $\bar{c}$).

2.5 Submanifolds

As the concepts listed here are essential only for the proofs of our results, we give only a brief summary: We assume that the reader is familiar with the concept of a smooth n-dimensional manifold N (an n-manifold). An embedded smooth m-submanifold M of N is characterized by the existence, for $p \in M$, of a local diffeomorphism which transforms an N-neighbourhood U of p to $\mathbb{R}^n$ and $U \cap M$ to $\mathbb{R}^m \subset \mathbb{R}^n$. An immersion is a smooth mapping whose differential is one-to-one (but not necessarily onto). Then locally the immersion is also one-to-one. An immersed k-submanifold is the image of a smooth k-manifold under an immersion. The difference between embedded and immersed submanifolds is therefore that the latter may have 'self-intersections', but neither are allowed to have 'singularities'.

An embedded submanifold M_1 and an immersed submanifold M_2 of N are transverse (we write $M_1 \pitchfork M_2$), if for all points $p \in M_1 \cap M_2$ the tangent spaces T_pM_1, T_pM_2 span T_pN. Then $M_1 \cap M_2$ is an immersed $(\dim M_1 + \dim M_2 - \dim N)$-dimensional submanifold of N, whose tangent space equals $T_pM_1 \cap T_pM_2$.

If a smooth mapping ϕ of a smooth m-manifold M into a smooth n-manifold N has constant rank r (i.e., at all points its differential's rank as of a linear mapping equals r), then $\phi(M)$ is a smooth immersed r-dimensional submanifold of N.

§3. Local Contact Situation

If a body is bounded by a smooth surface f, this surface has an inside and an outside. The unit surface normals can point to either side, depending on the parametrization. If two bodies, which are bounded by smooth surfaces f', f'', touch each other, the curvatures of f', f'' give information whether they intersect locally or not.

It should be remarked that some methods proposed in the literature (cf. [10,12]) for avoiding local intersections are only approximations, and one can find surfaces where they won't work. Also it is important to note that the presence or absence of local intersections is completely independent of the actual motion of the two bodies.

Definition. *The interior* $\mathrm{int}(i)$ *of a conic i centered in the origin is void if i is void, and otherwise is the connected component of $\mathbb{R}^2 \setminus i$ which contains* 0 *and whose boundary is i. Its exterior* $\mathrm{ext}(i)$ *is the complement of* $\mathrm{int}(i) \cup i$.

We assume that f', f'' are parametrized such that their unit normal vectors in the common point coincide, and that this common normal vector points to the inside for f' and to the outside for f''. We further assume that not all four principal curvatures are zero. We consider the Dupin indicatrices i'_+, i'_-, i''_+, and i''_- for f' and f'', respectively.

Theorem 3. *Under these assumptions, the two bodies in question intersect locally, if one of the intersections $i''_+ \cap \operatorname{int}(i'_+)$, $i''_- \cap \operatorname{ext}(i'_-)$ is not void. They do not intersect, if $i''_+ \cap i'_+$, $i''_+ \cap \operatorname{int}(i'_+)$, $i''_- \cap i'_-$, $i''_- \cap \operatorname{ext}(i'_-)$ are void.*

Proof: (Sketch) The theorem follows from the fact that a twice continuously differentiable surface may be approximated of second order by the graph of a quadratic function, whose contour lines are scaled versions of the Dupin indicatrices. □

Note that the theorem says nothing about the cases that all principal curvatures are zero, the indicatrices touch each other in two opposite points, or even coincide (cf. Fig. 1, right). In that case, second derivatives are not sufficient to decide if there are local intersections. In practice, this does not matter very much because the only case that is likely to occur with nonzero probability is that of a flat end mill shaping a planar surface, which does not have self-intersections.

In [8,16,23] it is shown how to define indicatrices in the case of *piecewise* curvature-continuous surfaces. The theorem is valid also in this more general case.

§4. Configuration Manifolds and their Tangent Spaces

For many problems concerning the milling of free-form surfaces, it is important to know the degree of freedom of a motion constrained in various ways. Typical constraints are: Motion by translations such that a surface remains in contact with another surface (3-axis milling), motion such that a surface remains in contact with another surface (possible set of tool positions in 5-axis milling), motion such that a milling tool remains in contact with a surface and its axis is contained in some previously prescribed line congruence (a possible way to do 5-axis milling). These topics will be discussed in Subsections 4.1–4.3.

4.1 Translational motions constrained by surface-surface contact

Consider two surfaces f', f'' which have a common point $p = f'(u') = f''(u'')$ and share a common unit surface normal $n' = n''$ there. Imagine the first surface moving by translations such that it always touches the second surface (i.e., at every instant t there are a translation vector $a(t)$ and parameter values $u'(t)$ and $u''(t)$ such that $f'(u'(t)) + a(t) = f''(u''(t))$). One would expect that this motion has two degrees of freedom, if we do not count intersections of the surfaces.

If $q(x_1, x_2) = \sum a_{ij} x_i x_j$ is a bivariate homogeneous quadratic polynomial in the variables x_1, x_2, we call the rank of the (2×2)-matrix a_{ij} the rank of q. The zero set of q consists of the entire plane in the case of zero rank, of one line if the rank is one, and of two real or two conjugate complex lines if the rank is two.

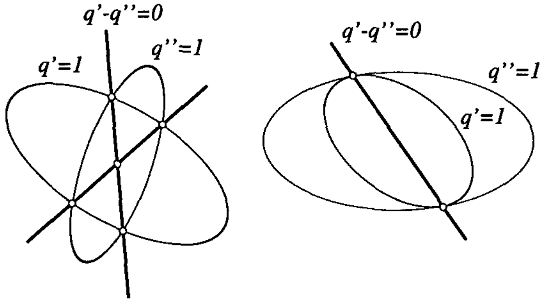

Fig. 1. Indicatrices $q' = 1$, $q'' = 1$ and zero set of $q' - q''$. Left: $\mathrm{rk}(q' - q'') = 2$, Right: $\mathrm{rk}(q' - q') = 1$. .

We have the following theorem (cf. [8,16,23]), which is valid for all contact situations where at least one surface has no parabolic or flat point, or both have parabolic points but the principal tangents do not coincide.

Theorem 4. *Write the equation of the oriented Dupin indicatrices of f', f'' at the contact point in the form $q'(x_1, x_2) = 1$ and $q''(x_1, x_2) = 1$, where x_1, x_2 are Cartesian coordinates in the tangent plane, and q', q'' are bivariate homogeneous quadratic polynomials in x_1, x_2. If the condition stated immediately before this theorem is satisfied, then the rank of $q' - q''$ gives the infinitesimal degree of freedom of translational motions constrained by the contact of f', f'' (see Fig. 1).*

Proposition. *If, under the assumptions of Th. 4, the infinitesimal degree of freedom is two, then so is the local degree of freedom.*

4.2 Motions constrained by surface-surface contact

Definition. *The set of proper Euclidean motions which transforms a surface f' such that it touches a surface f'' is called the* configuration space *$C = C(f', f'')$ of surface-surface contact.*

Clearly, a position $g \in C$ is not determined by the contact points alone, because we still may rotate f' about the contact normal. But if we prescribe a unit tangent vector $(p'; w')$ of f' and $(p'; w'')$ of f'' (which means $p' = f'(u_1', u_2')$ and $w' = df'(v')$ with $\|w'\| = 1$, and the same for (p'', w'')), then there is a unique Euclidean motion $g \in C$ which maps not only the point p' onto p'', but also the tangent vector w' onto w''. If we rotate both w', w'' about an angle ϕ_2 this leads to the same $g \in C$, so C is a smooth image of the factor manifold $\widetilde{C}$ of such equivalence classes of unit tangent vectors, which is five-dimensional.

This shows that we may expect five degrees of freedom, if f' moves under the constraint that it touches f'' in some point. The following theorem is given in [18,24]:

Theorem 5. *In the notation of Th. 4, the motion of a surface under the single constraint that it touches a second one, has five degrees of freedom if $q' - q''$ has rank two. If that is the case for all possible contact points, then the configuration space is an immersed five-dimensional submanifold of the motion group.*

Note that the theorem is valid without the additional assumption made in Th. 4 in case some principal curvatures are zero. It is easy to describe the tangent space T_gC of the configuration manifold at a contact position $g \in C \subset G$ (to be precise, the tangent space of the immersion described at the beginning of Sect. 4.2). The proof of the following proposition can be found in [18].

Proposition. *The linear space T_gC of infinitesimal motions which belong to paths in C is five-dimensional, and the axes of its infinitesimal rotations are the lines which intersect the contact normal.*

In a Cartesian coordinate system whose origin is the contact point and whose x_3-axis is the contact normal, T_gC has the equation $c_{03} = 0$.

4.3. Motions constrained by congruences

Here we consider the motion of a rigid body Σ such that a line a of Σ is contained in a congruence $\mathcal{K}$. This subset of the group G of proper Euclidean motions will be denoted by K. We assume that $\mathcal{K}$ is parametrized by $l(u_1, u_2) = p(u_1, u_2) + [v(u_1, u_2)]$.

If $g \in K$, then it is obvious that both $\tau \circ g$ and $\rho \circ g$ are in K, if τ is a translation parallel to $g(a)$ and ρ is a rotation with axis $g(a)$.

Lemma. *Assume that $\mathcal{K}$ is a smooth line congruence which is regular at $g(a)$. Then K is a four-dimensional smooth submanifold of G in a neighbourhood of g.*

Proof: (Sketch) Let $g \in K$ be a position of Σ such that $g(a) \in \mathcal{K}$. We compute K's tangent space T_gK of infinitesimal motions at g: Consider a curve $(u_1(t), u_2(t))$ in $\mathcal{K}$'s parameter domain such that $l(t) = l(u_1(t), u_2(t))$ is a ruled surface within $\mathcal{K}$ with $l(0) = g(a)$. The helical motion described in Sect. 2.4 is tangent to K. If we choose two such curves with linearly independent tangent vectors, this gives two linearly independent infinitesimal motions of T_gK, if $\mathcal{K}$ is regular at $g(a)$ (the proof of this is left as an exercise to the reader).

Obviously all infinitesimal translations parallel to $g(a)$ are in T_gK, and so are the infinitesimal rotations with axis $g(a)$. If $g(a)$ is the line $p + [v]$, then the former is described by the six-tuple $(0, v)$ and the latter by $(v, p \times v)$. Now T_gK is the linear span of these four infinitesimal motions. $\square$

In case that $g(a)$ is a hyperbolic line of $\mathcal{K}$, there is a simple geometric characterization of the infinitesimal rotations of T_gK:

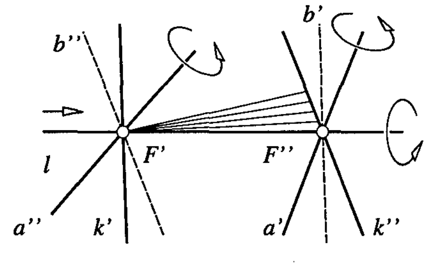

Fig. 2. Axes of infinitesimal rotations in K.

Proposition. *If $\mathcal{K}'$ is a hyperbolic linear congruence tangent to $\mathcal{K}$ at the line $g(a)$ and has axes k', k'', then the axes of infinitesimal rotations in T_gK form a hyperbolic linear congruence $\mathcal{L}$, whose axes b', b'' are incident with $g(a) \cap k''$, $g(a) \cap k'$, respecively, are orthogonal to $g(a)$, and are such that $k', b', g(a)$ as well as $k'', b'', g(a)$ are coplanar (see Fig. 2).*

Proof: It is easy to see that the lines a', a'', incident with $k' \cap g(a)$, $k'' \cap g(a)$ and orthogonal to k', k'', respectively, are axes of infinitesimal rotations contained in T_gK. The axis of the infinitesimal translations along $g(a)$ (which is the line at infinity orthogonal to $g(a)$), $g(a)$ itself, and a', a'' intersect both b', b''. We already know that the set of axes is a linear congruence; because of these four intersections the lines b', b'' are necessarily the axes of $\mathcal{L}$. □

The following is used later:

Proposition. *A point of $g(a)$ which contains two different axes of infinitesimal rotations of T_gK must be a focal point of $g(a)$.*

Proof: At least one of the two axes is not $g(a)$ itself, and the rotation about this axis transforms $g(a)$ into a line which intersects $g(a)$. Looking at the list of linear tangent congruences in Sect. 2.4 shows that this is only possible in a focal point. □

4.4. Multiple constraints

Assume that a rigid body Σ, bounded by a smooth surface f', moves such that f' remains in contact with a surface f'', and that in addition a line a of Σ is contained in a smooth line congruence $\mathcal{K}$.

With C being the configuration space of surface-surface contact (see Sect. 4.2) and K as in Sect. 4.3 the set of possible positions g of Σ is given by the intersection $C \cap K$. The following theorem shows under what circumstances $C \cap K$ is actually a smooth three-dimensional submanifold of positions, as is to be expected when comparing dimensions:

Theorem 6. *If T_gK is not contained in T_gC, then $C \cap K$ is a three-dimensional immersed submanifold in a neighbourhood of g. This is always the case if $g(a)$ is not parallel to the contact tangent plane.*

Proof: We have to show that $C \pitchfork K$. Because $\dim T_gC = 5$, this is always the case if T_gK is not contained in T_gC. If the contact tangent plane is not parallel to l, then any small translation parallel to l leaves C and therefore in this case $C \pitchfork K$, and $\dim(C \cap K) = \dim C + \dim K - \dim G = 3$. □

If Σ has rotational symmetry, and the line $g(a)$ which is forced to belong to $\mathcal{K}$ is its axis, then we can say more about the set $C \cap K$ of admissible positions. Clearly, any rotation about the axis does not change Σ, so $g \in C \cap K$ implies that $\rho \circ g \in C \cap K$ whenever ρ is such a rotation.

We choose a reference point p on the axis, and look at the three-parameter family of its positions, which is actually only a two-parameter family:

Proposition. *If Σ has rotational symmetry, a reference point p in Σ's axis a traces out a regular two-surface while undergoing all transformations of $C \cap K$, provided that p is never a focal point of $g(a)$. This path surface is transverse to l if l is not parallel to the contact tangent plane.*

Proof: We consider the mapping $\phi : g \mapsto g(p)$ of $C \cap K$ to $\mathbb{R}^3$. All infinitesimal rotations about $g(a)$ assign zero velocity to p, which implies that the rank of ϕ is not greater than two. If p has zero velocity also for other infinitesimal motions of $T_g(C \cap K)$, these must be infinitesimal rotations, and we can use the proposition at the end of Sect. 4.3 to conclude that the rank of ϕ is indeed two, and its image a smooth 2-surface. If l and the contact tangent plane are not parallel, then no infinitesimal translation parallel to l is in $C \cap K$, and the path surface cannot be parallel to l. □

§5. Movement of the Contact Point

It is important to study the dependency of the contact point on the contact positions. This will be done for two different types of constraints.

5.1. Motions constrained by surface contact

In Sect. 4.2 we stated that the configuration space $C(f', f'')$ of surface-surface contact is an immersed image of $\widetilde{C}$ as described in Sect. 4.2, if the Dupin indicatrices of f', f'' fulfill a certain condition (Th. 5).

If this is the case, then there is a local inverse $C \to M$, and so the contact point depends on the positions $g \in C$ in a smooth way locally. This (local) mapping will be denoted by ψ.

As $\dim C = 5$ and the contact point varies in a 2-surface, there is a 3-dimensional kernel subspace $\ker d_g\psi \subset T_gC$ of infinitesimal motions which do not (infinitesimally) change the contact point. The following is an easy exercise in differential geometry:

Lemma. *If f', f'' are two surfaces having contact at a point p, n is the unit normal vector of f', and w_1, w_2 are principal tangent vectors, corresponding to curvatures κ_1, κ_2, then all infinitesimal rotations about the axes $p + [n]$, $p + n/\kappa_1 + [w_2]$, $p + n/\kappa_2 + [w_1]$ are contained in the tangent space T_gC of the configuration manifold, and do not (infinitesimally) change the contact point on f''.*

If a principal curvature is zero, the corresponding axis will be at infinity. Obviously rotations of these three types span $\ker d_g\psi$.

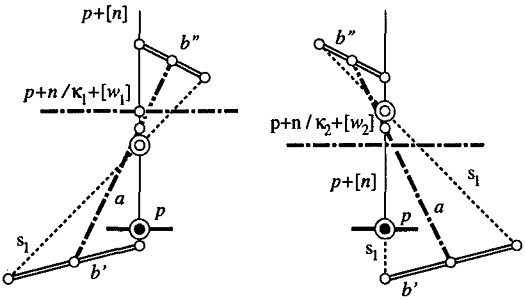

Fig. 3. Situation where the contact point has only one infinitesimal degree of freedom (cf. Section 5.2): front view and lateral view.

5.2. Multiply constrained motion

Now we consider the configuration manifold $C \cap K$ of a motion constrained by surface-surface contact and a line congruence $\mathcal{K}$ as described in Sect. 4.4. The contact point still depends smoothly on $g \in C \cap K$. But does it have nonzero velocity for all infinitesimal motions different from the rotations about the contact normal? Obviously, that depends on the intersection $\ker d_g\psi \cap T_g(C \cap K)$. Because $\ker d_g\psi \subset T_gC$, this intersection equals $\ker d_g\psi \cap T_gK$. Depending on its dimension, there are the following three possibilities:

- dim = 1: Only the infinitesimal rotations about the contact normal are in $\ker d_g\psi$. The infinitesimal motion of the contact point is two-dimensional.
- dim = 2: The rank of ψ is one, and the contact point varies infinitesimally only in one direction.
- dim = 3: $\mathrm{rk}\psi = 0$ and the contact point does not move infinitesimally.

If the line $g(a)$ belonging to the current position $g \in C \cap K$ is a hyperbolic line of $\mathcal{K}$, then the rank of ψ's restriction to $C \cap K$ can be determined geometrically (see Fig. 3):

Proposition. *We use the notation of the propositions in Sect. 4.3 and of the lemma in Sect. 5.1. If there are lines s_1, s_2 such that s_i intersects both b', b'', s_i is incident with $p+n/\kappa_i$ $(i = 1, 2)$, and $s_1 \subset p+[n]+[w_2]$, $s_2 \subset p+[n]+[w_1]$, then $\mathrm{rk}\psi|C \cap K \leq 1$, otherwise $\mathrm{rk}\psi|C \cap K = 2$.*

Proof: The lemma in Sect. 5.1 describes $\ker_g \psi =: A$ and the first proposition in Sect. 4.2 does the same for $T_gK =: B$. To compute $A \cap B$ we apply the duality which assigns to a linear space $A \subset \mathbb{R}^6$ the linear space $A^\star$ of those infinitesimal motions $(a^\star, \bar{a}^\star)$ which fulfill $a \cdot \bar{a}^\star + \bar{a} \cdot a^\star = 0$ for all $(a, \bar{a}) \in A$. Clearly, $\dim A^\star = 6 - \dim A$ and $(A \cap B)^\star$ is the linear span of $A^\star$, $B^\star$.

If is well known that an infinitesimal rotation is in $A^\star$ if and only if its axis intersects all the axes of the infinitesimal rotations of A. Thus, the rotation axes in $A^\star$ consist of two pencils with the same vertices as those of A, but with orthogonal planes; and the rotation axes of $B^\star$ are just the lines b', b''.

Rank ≤ 1 means $\dim(A^\star + B^\star) \leq 4$, or that the span of b' and $A^\star$ contains b''. It is easily seen that apart from even more special cases, this happens if the axes of this span are a hyperbolic linear congruence with axes s_1, s_2. □

§6. Collision Checking

It is possible that a milling tool has no local intersections with the finished surface, but while in contact with the surface at one point, it cuts into another part. Algorithms which test for this type of intersection of two bodies in space are time consuming, and therefore we want to circumvent the general collision test in some way. In some cases we are able to predict the total absence of collisions based only on the curvatures of the boundaries of the two bodies involved.

We say that a surface Φ is millable by a body Σ, if (a) there are no local intersections, and (b) it is possible for Σ to move, within previously imposed constraints, along the surface Φ such that it touches Φ during this motion, but never actually intersects it.

6.1. 3-axis milling

3-axis milling means that a milling tool Σ rotates about its axis a and moves such that a remains parallel to a fixed line, and Σ always touches the finished surface. As the rotation about a is not important for geometric considerations, we disregard it completely and consider a body which moves in a translational manner.

We assume that Σ as well as the workpiece Φ are bounded by piecewise twice continuously differentiable surfaces (convex edges are allowed). Then the so-called general offset surface (defined below, see Fig. 4, right) of Φ with respect to Σ shows in its singularities and self-intersections the singular positions of the motion and the collisions (cf. [15,16]).

Definition. *Choose a reference point p attached to Σ. Consider the set of translations τ such that $\tau(\Sigma)$ touches Φ (disregarding intersections). Then the set of all points $\tau(p)$ is called the* general offset surface *of Φ with respect to Σ.*

Theorem 7. *If Σ is strictly convex with positive principal curvatures, and Φ is connected, then the general offset surface of Φ with respect to Σ is smooth. It is regular in all points which correspond to translationally regular contact positions, and if it is both regular and free of self-intersections, then Φ is globally millable by Σ.*

Proof: (cf. [16,23]) For all points $p \in \Phi$ there is a unique position g such that $g(\Sigma)$ touches Φ in p. The parametrization of the general offset which thus is induced by the parametrization of Φ is easily seen to be smooth. There is always a point p_0 such that $\Sigma \cap \Phi$ consists of p_0 only. If Σ touches Φ in p, and thereby cuts into another part of Φ, then let Σ move such that the contact point follows a curve which joins p_0 with p. At the first time that the set

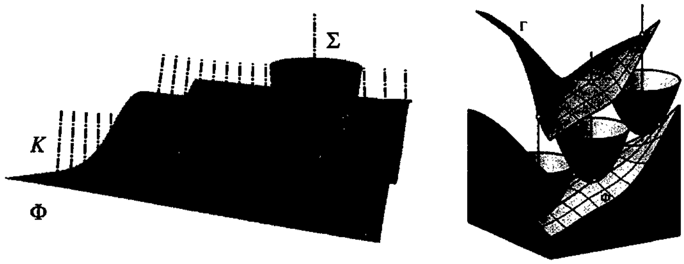

Fig. 4. Left: Line congruence $\mathcal{K}$, finished workpiece Φ and cutter Σ. Right: Workpiece Φ, milling-tool Σ and general offset surface Γ.

$\Sigma \cap \Phi$ consists of more than one point, Σ and Φ touch each other in all points of $\Sigma \cap \Phi$, which leads to a multiple point of the general offset. □

Thus, we have transformed the collision problem into the problem of determining singularities and self-intersections of certain surfaces. If Φ has some additional properties then it is not difficult to guarantee total absence of unwanted collisions provided that no local collisions occur:

- if Φ is convex and Σ, Φ are oppositely oriented (i.e., the body bounded by Φ is its inside); or
- if Φ is convex and Σ, Φ are equally oriented (i.e., the body bounded by Φ is its outside and Σ is inside); or
- if Φ is star-shaped and bounds its inside; or
- if Φ is star-shaped, bounds its outside, and Σ fits into the convex core of Φ (the convex core of a star-shaped set M is the set of points with respect to which M is star-shaped — it is a convex subset of M); or
- if Φ is the graph surface of a function over a planar domain whose boundary is millable by the 'top view' contour of Σ.

Proof: The proof of this can be found in [16], and generalizations actually unimportant for applications are studied in [23]. The idea of the proof, which is important also for the proof of Th. 8, is as follows: We assume that there is a 'projection' of entire space onto Φ. If e.g. Φ is convex, just choose any interior point o, and to project a point p, intersect the ray $\overrightarrow{op}$ with Φ.

Then consider the following mapping whose domain is Φ: For a point $x \in \Phi$, translate Σ such that it touches Φ there, and project a previously chosen reference point of Σ onto Φ. This mapping is shown to be smooth and orientation-preserving, and by scaling Σ with a factor λ $(1 \geq \lambda \geq 0)$ is deformed into the identity mapping. Differential topology allows now to conclude the Φ is one-to-one and onto, which means that no translate of Σ can touch Φ in more than one point. □

6.2. 5-axis milling constrained by a line congruence

Here 5-axis milling means that the milling tool moves such that it touches the finished surface, and its axis is always contained in a line congruence $\mathcal{K}$.

Definition. *A regularity domain of $\mathcal{K}$ is an open domain M in $\mathbb{R}^3$ such that $\mathcal{K}$ defines a fibration of M (every point of M is contained in exactly one set $l \cap M$, where l is a line of the congruence). A section of a regularity domain M is a surface which intersects all sets $l \cap M$ exactly once. The regularity domain M is a tubular neighbourhood of its section Φ if it is diffeomorphic to $\Phi \times \mathbb{R}$.*

The domain M is a tubular neighbourhood of Φ if the sets $l \cap M$ are open line segments, the point $p = \Phi \cap l \cap M$ is an interior point of this segment, and the initial and end points of the segment depend smoothly on p.

We say that a convex body Σ with rotational symmetry is admissible for a line congruence $\mathcal{K}$ and a connected closed surface Φ which is the boundary of a subset of $\mathbb{R}^3$ if the following is fulfilled:

- There is a regularity domain M for $\mathcal{K}$ which is a tubular neighbourhood of Φ.
- Σ moves such that it is entirely contained in the regularity domain.
- In no position of $C \cap K$ the contact tangent plane is parallel to the axis of Σ.
- The contact point has two infinitesimal degrees of freedom for all scaled versions $\lambda\Sigma$, $0 \le \lambda \le 1$ (cf. Sect. 5.2).

These conditions are actually easy to fulfill in practice except for the last one, which is difficult to detect in advance. In the special case of three-axis milling ($\mathcal{K}$ is a bundle of parallel lines), this requires that all contact points on Σ are elliptic surface points.

Suppose that we are given a surface Φ and a milling-tool Σ, and we have chosen a congruence $\mathcal{K}$, and have found, for all contact points on Φ, a position $g(\Sigma)$ such that $g(\Sigma)$'s axis is in $\mathcal{K}$. Suppose we have already tested for local millability and the admissibility conditions described above. Then we have the following

Theorem 8. *Under the circumstances described above, the cutter does not interfere with the surface Σ, i.e., Φ is globally millable by Σ.*

Note that in many cases it will be sufficient to check the admissibility only for the cutter head, because collisions of the cutter shaft with the workpiece will be treated by different methods (see the introduction)

Proof: In Section 5 we have established that the contact point depends smoothly on Σ's position g. Let a be Σ's axis, and let ρ be a rotation about $g(a)$. Clearly $\rho \circ g$ is again a contact position with the same contact point as g. If we choose a reference point p on Σ's axis and inside Σ, the path surface of p also depends smoothly on g, and $\rho \circ g(p) = g(p)$.

This means that the contact point depends smoothly on the position $g(p)$ of the reference point. The last admissibility condition ensures that also $g(p)$ depends smoothly on the contact point. Thus, we can define a smooth mapping $f : \Phi \to \Phi$ as follows: A contact point $q \in \Phi$ is mapped to the corresponding point $g(p)$, which is subsequently mapped to the intersection

$f(q)$ of $g(a)$ with Φ. If we scale Σ with a factor λ ($0 \leq \lambda \leq 1$), we get mappings f_λ, where f_0 is the identity mapping of Φ onto itself, and $f_\lambda(q)$ depends continously on λ. f_λ is never singular since we are in a regularity domain, $f_1 = f$ is orientation-preserving because f_0 is, and the number of pre-images of a point is the same for $f_0 = \mathrm{id}$ and $f_1 = f$. This shows that f is one-to-one and onto. If $g(\Sigma)$ touches Φ in one point and cuts into Φ in another, there also is a position $g(\Sigma)$ where Σ touches in two different points (cf. the proof of Th. 7). These two points have the same f-image by definition of f, which contradicts bijectivity of f. $\square$

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